CLASSIFICATION OF LAGRANGIAN FIBRATIONS OVER A KLEIN BOTTLE

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ABSTRACT. This paper completes the classification of regular Lagrangian fibrations over compact surfaces. [13] classifies regular Lagrangian fibrations over $\mathbb{T}^2$. The main theorem in [8] is used to in order to classify integral affine structures on the Klein bottle K^2 and, hence, regular Lagrangian fibrations over this space.

CONTENTS

1. INTRODUCTION

This paper classifies regular Lagrangian fibrations with compact fibre whose base space is a Klein bottle K^2.

Definition 1.1. A fibration $F \overset{\iota}{\hookrightarrow} M \overset{\pi}{\to} B$ is *Lagrangian* if the total space M is a symplectic manifold and each fibre F is a Lagrangian submanifold.

The most elementary examples of Lagrangian fibrations arise from Hamiltonian dynamics, where the natural setting is the cotangent bundle T^*B of a manifold B with canonical symplectic form ω_0. The bundle $\mathbb{R}^n \hookrightarrow (\mathrm{T}^*B, \omega_0) \to B$ is a Lagrangian fibration. [13] classifies regular Lagrangian fibrations with fibre $\mathbb{R}^n$ subject to the constraint that the fibre be *complete*. More examples come from completely integrable Hamiltonian systems.

Definition 1.2. A *completely integrable Hamiltonian system* on a symplectic manifold (M^{2n}, ω) is a set of n functions $f_1, \ldots, f_n : M \to \mathbb{R}$ satisfying the following conditions

- The functions are in involution - i.e. $\{f_i, f_j\} = 0$ for all i, j, where $\{.,.\}$ is the Poisson bracket induced by the symplectic form ω;
- $\mathrm{d}f_1 \wedge \ldots \wedge \mathrm{d}f_n \neq 0$ almost everywhere.

The local properties of a completely integrable Hamiltonian system near a regular fibre of the map $\mathbf{f} = (f_1, \ldots, f_n) : M \to \mathbb{R}^n$ are well-known, as they are completely described by the Liouville-Arnol'd-Mineur theorem (see [1] for a proof).

Theorem 1.1 (Liouville-Arnold-Mineur)**.** *Let* $\mathbf{f} = (f_1, \ldots, f_n) : M \to \mathbb{R}^n$ *be a completely integrable Hamiltonian system with n degrees of freedom and let* $x \in \mathbb{R}^n$ *be a regular value in the image of* $\mathbf{f}$*. Suppose that* $\mathbf{f}^{-1}(x)$ *has a compact, connected component, denoted by* F_x*. Then*

- F_x *is a Lagrangian submanifold of* M *and is diffeomorphic to* $\mathbb{T}^n$*;*
- *there is a neighbourhood* V *of* F_x *in* M *that is symplectomorphic to an open neighbourhood* W *of* $\mathbb{T}^n$ *in* $\mathrm{T}^*\mathbb{T}^n$*, as shown in the diagram below.*

Clearly, $W \cong D^n \times \mathbb{T}^n$*. Let* $\tilde{\varphi}(p) = (a^1(p), \ldots, a^n(p), \alpha^1(p), \ldots, \alpha^n(p))$*. The coordinates* a^i *are called the* **actions** *and depend smoothly on the functions* f_i*. The coordinates* α^i *are called the* **angles***.*

In particular, in a neighbourhood of a regular value of $\mathbf{f}$ a completely integrable Hamiltonian system is given by a regular Lagrangian fibration. Conversely, given a regular Lagrangian fibration it is given locally by a completely integrable Hamiltonian system ([7]); Theorem 1.1 therefore shows that the fibre of a regular Lagrangian fibration with compact fibre is a torus $\mathbb{T}^n$.

A natural question to ask is whether for a completely integrable Hamiltonian system the local action-angle coordinates given by Theorem 1.1 can be extended to global coordinates. In [7] Duistermaat constructed obstructions to the existence of such global coordinates for a regular Lagrangian fibration with compact fibres. These obstructions are of a topological nature and are

- the *monodromy* - obstruction for the fibration to be a principal $\mathbb{T}^n$-fibration (Theorem 1.1 shows that any Lagrangian fibration is locally a principal $\mathbb{T}^n$-fibration);
- the *Chern class* - obstruction to the existence of a global section.

His motivation came from Cushman's work on the spherical pendulum, where monodromy naturally arises [3]. In [16] Zung introduced a *symplectic* invariant, called the *Lagrangian class*, which is the obstruction to the existence of a fibrewise symplectomorphism between a given Lagrangian fibration and an appropriately defined reference Lagrangian fibration. His work is a natural continuation of the study of isotropic fibrations carried out by Dazord and Delzant in [6].

Definition 1.3. An *integral affine structure* on a manifold B is a choice of atlas for B in which all the transition functions lie in the group $\mathrm{Aff}_{\mathbb{Z}}(\mathbb{R}^n) = \mathrm{GL}(n, \mathbb{Z}) \ltimes \mathbb{R}^n$, where multiplication is defined as

$$(A, x) \cdot (B, y) = (AB, x + Ay)$$

The base space B of a Lagrangian fibration carries a natural integral affine structure [2, 4, 7, 16]. This observation connects regular Lagrangian fibrations with the

theory of (integral) affine geometry, which is a very interesting and rich field of mathematics and has been extensively studied [8, 12, 15]. There are topological restrictions on the topology of integral affine manifolds [12]; this paper shows that the only compact surfaces that admit integral affine structures are the 2-torus $\mathbb{T}^2$ and the Klein bottle K^2. This is tantamount to saying that these are the only compact surfaces that can be the base space of regular Lagrangian fibrations (see Lemma 2.2 below). There are examples of regular Lagrangian fibrations whose base space is a non-compact 2-manifold (e.g. the annulus or an open Möbius strip).

[13] classifies regular Lagrangian fibrations whose base space is $\mathbb{T}^2$; this is obtained using the main result from [8] and exploiting techniques developed in [14]. This paper classifies regular Lagrangian fibrations whose base space is a Klein bottle K^2, therefore completing the classification of regular Lagrangian fibrations over compact surfaces. The classification is up to a fibrewise symplectomorphism covering the identity on the base space; there is no loss of generality in imposing this condition, since there exists a fibrewise symplectomorphism

$$
\begin{array}{ccc}
M & \xrightarrow{\;F\;} & M' \\
\downarrow & & \downarrow \\
B & \xrightarrow{\;f\;} & B
\end{array}
$$

covering a diffeomorphism $f : B \to B$ on the base space if and only if there is a fibrewise symplectomorphism

$$
M \xrightarrow{\;\;\;\;\Phi\;\;\;\;} f^*M'
$$
$$
\searrow \qquad \swarrow
$$
$$
B
$$

covering the identity on B. Furthermore this approach clarifies the importance of the integral affine structure on the base space B. Henceforth it is understood that by 'classification of regular Lagrangian fibrations' we mean up to fibrewise symplectomorphisms covering the identity on the base space unless otherwise stated.

The structure of the paper is as follows. Section 2 discusses generalities of Lagrangian fibrations and affine manifolds, defining the invariants of the fibrations and highlighting the interplay between the affine structure on the base manifold B and the monodromy of the fibration. The main result of this section is that any integral affine Klein bottle is *geodesically complete*, which is a consequence of the main theorem in [8]. This result is crucial to the rest of the present paper and is used in section 3 to classify all possible integral affine structures on K^2 using elementary methods. Section 4 constructs all possible examples of regular Lagrangian fibrations $F \hookrightarrow M \longrightarrow K^2$, using ideas from [14].

2. Generalities on Lagrangian Fibrations and Affine Manifolds

Throughout this paper a 'Lagrangian fibration' is regular with compact fibres whose base space B is connected unless otherwise stated. This section briefly goes through the topological and symplectic invariants of Lagrangian fibrations and then relates the monodromy to the integral affine structure on the base space B of a

Lagrangian fibration. [4, 7, 13, 16] provide more details about the construction of these invariants.

2.1. **Invariants of Lagrangian Fibrations.** Let us begin with the archetypal Lagrangian fibration with compact fibre. Let B be a manifold and let (T^*B, ω_0) be its cotangent bundle with canonical symplectic form ω_0. Let $P \to B$ be a discrete subbundle of maximal dimension of the cotangent bundle such that $P \to B$ is locally spanned by closed forms. There are several names in the literature for this discrete subbundle; *period lattice* or *period-lattice bundle* are the most common. The symplectic form ω_0 descends to a well-defined symplectic form on the quotient $T^*B/P \to B$ (which, by abuse of notation, is denoted by ω_0 too).

Definition 2.1. The topological fibration $T^*B/P \to B$ is the *topological reference Lagrangian fibration with period-lattice bundle P*.

For a given manifold B there may be many non-isomorphic topological reference Lagrangian fibrations; the isomorphism type of the bundle depends on the choice of period-lattice bundle as the next example illustrates [13].

Example 2.1. Let $T^*\mathbb{T}^2$ be the cotangent bundle of the 2-torus with canonical symplectic form. Then there exist period lattice bundles $P, P' \to \mathbb{T}^2$ such that the quotients $T^*\mathbb{T}^2/P$, $T^*\mathbb{T}^2/P'$ are diffeomorphic to $\mathbb{T}^4$ and to a Thurston-type manifold respectively. These two spaces are not homeomorphic since their first Betti numbers are distinct.

Classifying integral affine structures on the Klein bottle allows to classify the distinct isomorphism types of admissible period-lattice bundles and, hence, to classify Lagrangian fibrations over K^2. Fix a Lagrangian fibration $F \overset{\iota}{\hookrightarrow} M \overset{\pi}{\longrightarrow} B$ and consider the natural map $\rho : \pi_1(B) \to \mathrm{Aut}(F) \cong \mathrm{GL}(n, \mathbb{Z})$ associated to the fibration.

Definition 2.2. ρ is called the *monodromy* of the Lagrangian fibration.

Remark 2.1. The map ρ is only defined up to conjugacy in $\mathrm{GL}(n, \mathbb{Z})$, since it depends upon the choice of basepoint $b \in B$. The conjugacy class of ρ is called the *free monodromy* of the fibration; it determines the isomorphism class of the period lattice bundle $P \to B$ associated to the fibration.

Any topologically trivial Lagrangian fibration has a global section (the image of the zero section $s_0 : B \to T^*B$ under the quotient is an example). Given a Lagrangian fibration $F \overset{\iota}{\hookrightarrow} M \overset{\pi}{\longrightarrow} B$ with monodromy map $\rho : \pi_1(B) \to \mathrm{GL}(n, \mathbb{Z})$, a natural question to ask is whether it admits a global section $s : B \to M$. Consider the following diagram

$$
\begin{array}{ccc}
 & & M \\
 & {\scriptstyle s}\nearrow & \downarrow {\scriptstyle \pi} \\
B & \xrightarrow{\mathrm{id.}} & B
\end{array}
$$

where id. $: B \to B$ denotes the identity map. Since the fibre F is simple, obstruction theory shows that the obstruction to the existence of a global section is an element $c \in \mathrm{H}^2(B; \mathbb{Z}_\rho^n)$.

Definition 2.3. c is the *Chern class* associated to the Lagrangian fibration.

The Chern class of Definition 2.3 is just the generalisation of the Chern class for principal $\mathbb{T}^n$-bundles. The following lemma is proved in [7].

Lemma 2.1. *Let* $F \overset{\iota}{\hookrightarrow} M \overset{\pi}{\longrightarrow} B$ *be a Lagrangian fibration with monodromy* $\rho : \pi_1(B) \to \mathrm{GL}(n, \mathbb{Z})$. *This fibration is isomorphic to the topological reference Lagrangian fibration* $\mathrm{T}^*B/P \to B$ *(where* $P \to B$ *is the period lattice bundle associated to the given fibration) if and only if the Chern class vanishes.*

It turns out that the (free) monodromy and Chern class of a Lagrangian fibration completely describe the *topological* type of the fibration, *i.e.* these invariants are topologically sharp. It is natural to try to carry this point of view over to the symplectic classification (the classification of fibrewise symplectomorphisms) of such fibrations. There are however some additional subtleties to take into account, which have to do with the concrete choice of period lattice bundle $P \to B$ as the following examples show.

Example 2.2. This example is due to Lukina [11]. Let (x, y) be coordinates on T^*S^1, where y is the coordinate along the fibre. Define period lattice bundles P and P' by

$$P = \{(x, y) \in \mathrm{T}^*S^1 \,|\, y \in \mathbb{Z}\} \qquad P' = \{(x, y) \in \mathrm{T}^*S^1 \,|\, y \in 2\mathbb{Z}\}$$

Consider Lagrangian fibrations $\mathrm{T}^*S^1/P \to B$ and $\mathrm{T}^*S^1/P' \to B$ where the symplectic forms are obtained from the canonical symplectic form $\omega_0 = \mathrm{d}x \wedge \mathrm{d}y$ on T^*S^1. The symplectic volumes of T^*S^1/P and T^*S^1/P' are 1 and 2 respectively and so these spaces cannot be symplectomorphic.

Example 2.3. This example elaborates on the previous one. The idea of Example 2.2 is to change the volume of the fibre, which is tantamount to asking that the two period lattice bundles $P \to S^1$ and $P' \to S^1$ do not have the same integral affine invariants. The volume is clearly one such invariant and it is a symplectic invariant. However, the following shows that it is not a *complete* invariant. Let $\mathrm{T}^*\mathbb{T}^2$ be the cotangent bundle of $\mathbb{T}^2$ with coordinates $(\mathbf{x}, \mathbf{y})$ (where $\mathbf{x}$ are coordinates on $\mathbb{T}^2$) and canonical symplectic form. Define two period-lattice bundles Q and Q' by

$$Q = \{(\mathbf{x}, \mathbf{y}) \in \mathrm{T}^*\mathbb{T}^2 \,|\, \mathbf{y} \in \mathbb{Z} \times \mathbb{Z}\} \qquad Q' = \{(\mathbf{x}, \mathbf{y}) \in \mathrm{T}^*\mathbb{T}^2 \,|\, \mathbf{y} \in 2\mathbb{Z} \times \frac{1}{2}\mathbb{Z}\}$$

The resulting Lagrangian fibrations $\mathrm{T}^*\mathbb{T}^2/Q \to \mathbb{T}^2$ and $\mathrm{T}^*\mathbb{T}^2/Q' \to \mathbb{T}^2$ cannot be fibrewise symplectomorphic (this can be seen in local coordinates), but the fibres have the same volume by construction.

The above examples show that isomorphic period-lattice bundles $P, P' \to B$ that are not integral-affinely isomorphic (*i.e.* the isomorphism on the fibre cannot be expressed by an integral matrix which is invertible over the integers) yield symplectically *distinct* symplectic reference Lagrangian fibrations. Thus the definition of a symplectic reference Lagrangian fibration with given period-lattice bundle is subtle as the explicit choice of period-lattice bundle (up to its integral affine invariants) matters. Let a fibrewise symplectomorphism covering the identity be given

$$M \overset{\Phi}{\longrightarrow} M'$$
$$\searrow \qquad \swarrow$$
$$B$$

and fix an explicit choice of period-lattice bundle $P \to B$ for the Lagrangian fibration $M \to B$. Then a local calculation in action-angle coordinates shows that a necessary condition for a fibrewise symplectomorphism covering the identity to exist is that the period-lattice bundle $P' \to B$ associated to $M' \to B$ is precisely the period lattice bundle $P \to B$, whereby 'precisely' indicates that the explicit choices of period lattice bundles $P, P' \to B$ be the same. In the rest of this paper, we fix an explicit choice of the period-lattice bundle $P \to B$ thereby bypassing the above difficulties.

Definition 2.4. The fibration $(\mathrm{T}^*B/P, \omega_0) \to B$ is the *symplectic reference Lagrangian fibration with period-lattice bundle* P

Having fixed a choice of symplectic reference Lagrangian fibration, it is possible to tackle the symplectic classification. A symplectic reference Lagrangian fibration always admits a global Lagrangian section, given by the image of the zero section $s_0 : B \to \mathrm{T}^*B$ under the quotient. The Lagrangian class measures the obstruction to the existence of such a section. The next example shows how in general the notion of isomorphism in the symplectic sense is distinct from the same notion in the topological category.

Example 2.4. Let $B = \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ be the torus with standard integral affine structure. Let (x, y) be local coordinates (x, y) and consider the Lagrangian fibration $\pi : M = \mathbb{T}^2 \times \mathbb{T}^2 = \mathrm{T}^*\mathbb{T}^2/P \to B$ where $P \to B$ is given by the span of the (globally defined) 1-forms $\mathrm{d}x$, $\mathrm{d}y$. Let ω_0 be the symplectic form on M arising from the standard quotient construction outlined above and let $\omega_\alpha = \omega_0 + \pi^*\phi_\alpha$, where $\phi_\alpha = \alpha \mathrm{d}x \wedge \mathrm{d}y \in \mathrm{H}^2(\mathbb{T}^2; \mathbb{R})$. It follows from the classification in [13] that the bundles $(M, \omega_0) \to B$ and $(M, \omega) \to B$ are fibrewise symplectomorphic (over the identity map on B) if and only if $\alpha \in \mathbb{Z}$. In particular, $\alpha \bmod 1$ determines the class of the fibration up to fibrewise symplectomorphism.

A description of the Lagrangian class in full generality can be found in [6, 16]. For the purpose of this paper, it suffices to notice that, for fixed monodromy representation ρ and Chern class $c \in \mathrm{H}^2(B; \mathbb{Z}^n_\rho)$, the space of symplectically distinct Lagrangian fibrations is a quotient of $\mathrm{H}^2(B; \mathbb{R})$. In the case of the Klein bottle K^2 we have $\mathrm{H}^2(K^2; \mathbb{R}) = 0$ and so there are no symplectic invariants to be considered in the classification of Lagrangian fibrations whose base space is K^2.

2.2. **Integral Affine Manifolds.** This subsection relates the integral affine structure inherited by the base space B of a Lagrangian fibration to the monodromy of the Lagrangian fibration itself. In particular, the linear part of the integral affine structure completely determines the monodromy and so, in order to classify the possible monodromies of Lagrangian fibrations with base space B, it is enough to classify the possible integral affine structures on B. The reader is referred to [8] and the references therein for more details about affine geometry.

The following basic lemma is the starting point.

Lemma 2.2. *A manifold B is the base space of a Lagrangian fibration if and only if it is an integral affine manifold.*

Showing that the base space of a Lagrangian fibration inherits an integral affine structure is an instructive exercise in dealing with Hamiltonian $\mathbb{T}^n$-actions, while

the other direction is just a computation in local coordinates to make sure that the symplectic form defined locally is actually a global form. For more details, see [4, 7, 16].

Given an integral affine manifold B, consider the associated *affine monodromy* representation $\lambda : \pi_1(B, b) \to \mathrm{Aff}_{\mathbb{Z}}(\mathbb{R}^n)$ [8], where $b \in B$ is a basepoint. Composing with the projection $\mathrm{Aff}_{\mathbb{Z}}(\mathbb{R}^n) \to \mathrm{GL}(n, \mathbb{Z})$, obtain the *linear monodromy* representation $\tilde{\rho} : \pi_1(B, b) \to \mathrm{GL}(n, \mathbb{Z})$. These maps depend on the choice of basepoint $b \in B$, but their conjugacy classes do not. $\tilde{\rho}$ is the linear monodromy of the flat, torsion free connection ∇ that B inherits from the integral affine structure. It is well-known that the linear monodromy of the integral affine structure induced on the base space of a Lagrangian fibration B is equal to the monodromy of the fibration in the sense of Definition 2.2 [6, 7, 16]. This approach is very helpful, for instance, to see that trivial monodromy implies that the Lagrangian fibration is a principal $\mathbb{T}^n$-bundle. Locally a Lagrangian fibration is a principal torus bundle. If the monodromy is trivial, then so is the linear holonomy of ∇; parallel transport the free and transitive local action using ∇ to define a globally free and transitive $\mathbb{T}^n$-action.

2.3. **Affine Structures on K^2.** In order to classify Lagrangian fibrations whose base space is K^2 it is necessary to classify the integral affine structures that K^2 admits. While this is a hard task for dimensions greater than 2, it is a manageable problem for surfaces. [13] classifies Lagrangian fibrations whose base space is $\mathbb{T}^2$; a key element in the proof is the main result in [8], briefly outlined below. Let B be a compact, affine n-dimensional manifold with nilpotent fundamental group. Then a natural question to ask is what the possible affine universal covers $\tilde{B}$ of B are. Recall that there exists a local diffeomorphism $D : \tilde{B} \to \mathbb{R}^n$ called the *developing map* [9].

Definition 2.5. An affine manifold B is *complete* if its developing map $D : \tilde{B} \to \mathbb{R}^n$ is a homeomorphism

Throughout this paragraph B is compact and its fundamental group $\pi_1(B)$ is nilpotent. Compactness does not imply completeness as shown in [9]. However, if B admits a *parallel volume form*, *i.e.* a non-zero exterior volume form on $\mathbb{R}^n$ which is invariant under the induced action of the linear holonomy group, then B is complete. These two properties are equivalent as shown in [8]. It is important to remark the importance of the parallel volume form geometrically. Its existence means that there are no expansions in the linear holonomy group and this implies that the linear part of the affine action of $\pi_1(B)$ on $\mathbb{R}^n$ is actually *unipotent*. This last implication relies heavily on the assumption that $\pi_1(B)$ is nilpotent and that the manifold B is compact. The following simple example shows that the result fails for non-compact manifolds with abelian fundamental group.

Example 2.5. Let $Y = \mathbb{R}^2 - \{0\}$, let $\pi = \pi_1(Y)$ denote its fundamental group and endow Y with the integral affine structure arising from the inclusion $Y \hookrightarrow \mathbb{R}^2$. Consider the following action

$$\pi \to \mathrm{GL}(2, \mathbb{Z})$$

(1)
$$\zeta \mapsto \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

where ζ denotes a fixed generator of π. This action corresponds to considering the involution $-I : Y \to Y$. The above action is free and properly discontinuous, so that $Y/\langle -I \rangle$ is an integral affine manifold with linear (and affine) holonomy map given by (1). Note that $Y/\langle -I \rangle \cong Y$ and thus the above is an example of a non-unipotent integral affine structure on a non-compact manifold Y with abelian fundamental group.

A compact orientable integral affine manifold has a parallel volume form, since the linear holonomy lies entirely in $\mathrm{SL}(n, \mathbb{Z})$. [8] proves the following

Theorem 2.1. *Any compact orientable integral affine manifold B with nilpotent fundamental group is complete and the linear holonomy is unipotent.*

In particular the following corollary also holds and is proved in [8]

Corollary 2.1. *Any integral affine structure on $\mathbb{T}^n$ is complete.*

Theorem 2.2 below shows how to extend Theorem 2.1 to a more general family of integral affine manifolds, namely those obtained as a quotient of a compact integral affine manifold with nilpotent fundamental group by some integral affine action.

Theorem 2.2. *Let B be a compact integral affine manifold and suppose that there exists a regular covering $\hat{B} \to B$ of B by a compact orientable integral affine manifold $\hat{B}$ whose fundamental group is nilpotent. Then any integral affine structure on B is complete.*

Proof. Fix an integral affine structure on B and pull it back to $\hat{B}$ via the regular covering $\hat{B} \to B$. Let X be the integral affine universal cover of B such that $D : X \to \mathbb{R}^n$ induces the given integral affine structure. There is a commutative diagram

$$
\begin{array}{ccc}
 & & X \\
 & \swarrow & \downarrow \\
\hat{B} & & \\
 & \searrow & \downarrow \\
 & & B
\end{array}
$$

which commutes by naturality of the pullback of integral affine structures. In particular it shows that X is an integral affine universal cover for both $\hat{B}$ and B and $\hat{B}$ satisfies the hypothesis of Theorem 2.1. Hence X is homeomorphic to $\mathbb{R}^n$ with the standard integral affine structure, proving completeness of B. $\qquad \square$

Corollary 2.2. *Any integral affine structure on K^2 is complete.*

3. Classification of Integral Affine Structures on K^2

Corollary 2.2 simplifies the classification of integral affine structures on K^2. Let Γ denote the fundamental group of K^2; it follows from Corollary 2.2 that the classification of integral affine structures on K^2 amounts to classifying all injective homomorphisms

$$\Gamma \to \mathrm{Aff}_{\mathbb{Z}}(\mathbb{R}^2)$$

inducing a free and properly discontinuous action of Γ on $\mathbb{R}^2$. Let Γ have presentation $\Gamma = \langle a, b \, | \, aba = b \rangle$ and let $G \lhd \Gamma$ be the abelian subgroup generated by a, b^2.

This subgroup corresponds to the fundamental group of the two-fold cover $\mathbb{T}^2$ of K^2. By abuse of notation denote the images of the generators a, b of Γ in $\mathrm{Aff}_{\mathbb{Z}}(\mathbb{R}^2)$ by a and b respectively. Write $a = (A, \mathbf{x})$ and $b = (B, \mathbf{y})$, where $A, B \in \mathrm{GL}(2, \mathbb{Z})$ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$. The following facts are true:

- since a, b^2 are generators for $\pi_1(\mathbb{T}^2)$, then $\det A = 1 = \det B^2$;
- b is an orientation reversing transformation and hence $\det B = -1$;
- the action of Γ on $\mathbb{R}^2$ is free, so that $\mathbf{x}, \mathbf{y} \neq 0$, as otherwise the origin is a fixed point of the action;
- the action given by the composite $\pi_1(\mathbb{T}^2) \hookrightarrow \Gamma \longrightarrow \mathrm{Aff}_{\mathbb{Z}}(\mathbb{R}^2)$ is also free and properly discontinuous since it defines the integral affine structure on $\mathbb{T}^2$ obtained by pulling back a fixed integral affine structure on K^2.

The following two basic lemmas are extremely useful and are proved in [13].

Lemma 3.1. *If $(C, z) \in \mathrm{Aff}_{\mathbb{Z}}(\mathbb{R}^2)$ has no fixed point in $\mathbb{R}^2$, then the linear transformation C has 1 as one of its eigenvalues.*

Lemma 3.1 is sometimes referred to in the literature as 'Hirach's principle' [8]. In some sense, the work done in [8] generalises the above lemma to the case when the affine manifold is compact and has nilpotent fundamental group.

Lemma 3.2. *For a matrix D having 1 as an eigenvalue, if $\det D = 1$ we have that $\mathrm{Tr}D = 2$ and if $\det D = -1$ then $\mathrm{Tr}D = 0$*

By Lemma 3.2 the transformation B satisfies $\det B = -1$ and $\mathrm{Tr}B = 0$. What matters in the following classification is simply the conjugacy class of B in $\mathrm{GL}(2, \mathbb{Z})$. There are only two such conjugacy classes and, within each class, we are free to choose whichever representative simplifies calculations the most. Take the following as initial representatives for the two conjugacy classes, following [14]

$$B_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad B_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

It remains to determine the matrices $A \in \mathrm{GL}(2, \mathbb{Z})$ with $\det A = 1$, $\mathrm{Tr}A = 2$ satisfying the relation $ABA = B$ where B is one of B_1, B_2 in 3. An algebraic calculation shows that the only possibilities are as follows

$$A = I, \quad B_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$A = A_1 = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} 1 & 0 \\ p & 1 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$A = I, \quad B_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$A_3 = \begin{pmatrix} 1+n & n \\ -n & 1-n \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$A_4 = \begin{pmatrix} 1-n & n \\ -n & 1+n \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

where $p, n \neq 0$.

In order to determine the translation components of a and b, impose the relation $aba = b$ once the linear part has been determined. It is possible to conjugate the generating set a, b by elements in $\mathrm{Aff}_{\mathbb{Z}}(\mathbb{R}^2)$ and also to switch from generators a, b to any other generating set for Γ satisfying the group relation. Let $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$ denote the translation components of a and b respectively. The two-fold covering $\mathbb{T}^2 \to K^2$ induces an injective homomorphism $\pi_1(\mathbb{T}^2) \to \pi_1(K^2)$; the composite map $\pi_1(\mathbb{T}^2) \to \pi_1(K^2) \to \mathrm{Aff}_{\mathbb{Z}}(\mathbb{R}^2)$ yields an integral affine structure on $\mathbb{T}^2$, which is generated by a and b^2 as above. When the linear part of a is just the identity, then the translation components of a, b^2 need define a lattice for the action to be free. In particular, the matrix

$$\begin{pmatrix} x_1 & 2y_1 \\ x_2 & 0 \end{pmatrix}$$

is non-degenerate and so $x_2, y_1 \neq 0$. Furthermore, the translation component of b^2 is non-trivial. It is necessary to check freeness of the action $\Gamma \to \mathrm{Aff}_{\mathbb{Z}}(\mathbb{R}^2)$. To this end, it is useful to note that words in Γ can be reduced to words of the form $a^q b^k$ for $k, q \in \mathbb{Z}$. In what follows, the explicit calculations appear in the first case only; all other cases are similar in spirit but the calculations are more cumbersome. Each case is dealt with separately and the corresponding generating set for the integral affine structure on the two-fold cover $\mathbb{T}^2$ of K^2 is also indicated. Throughout the rest of the section, let $\mathbf{e}_1, \mathbf{e}_2$ denote the standard basis of $\mathbb{R}^2$.

Case $A = I$, $B = B_1$. The group relation on the translation components reduces in this case to $B_1 \mathbf{x} = -\mathbf{x}$. Hence the translation component of a has to lie in the -1-eigenspace of the linear part of B and $x_1 = 0$. Conjugating both a and b by $(I, \frac{y_2}{2}\mathbf{e}_2)$ assume that $y_2 = 0$. Changing generators to a^{-1}, b^{-1} if necessary, assume further that $x_2, y_1 > 0$. Hence the generating set for these integral affine structures are of the form

(2) $a = (I, x\mathbf{e}_2), \qquad b = (B_1, y\mathbf{e}_1)$

where $x, y > 0$. It remains to check that this action is free. It is enough to consider the cases where the word in the group is of the form $a^q b^{2k+1}$ for $k, q \in \mathbb{Z}$, since the case $a^q b^{2k}$ follows from [13]. Then

$$a^q = (I, qx\mathbf{e}_2), \qquad b^{2k+1} = (B_1, (2k+1)y\mathbf{e}_1)$$

for all $k, q \in \mathbb{Z}$. For the action to be free we want there to be no solution to $(a^q b^{2k+1}) \cdot \mathbf{z} = \mathbf{z}$ for $\mathbf{z} \in \mathbb{R}^2$. If this equation has a solution then $0 = -(2k + 1)y$, which is absurd since $y \neq 0$. Hence the action is free. The induced structure on $\mathbb{T}^2$ is

$$a = (I, x\mathbf{e}_2), \qquad b^2 = (I, 2y\mathbf{e}_1)$$

where $x, y > 0$.

Case $A = A_1$, $B = B_1$. First, since $a^{-1}ba^{-1} = b$, it is possible to change the generating set from a, b to a^{-1}, b. Thus, switching a with a^{-1} if necessary, assume that $n > 0$. Conjugating a, b by $(I, -\frac{x_1}{n}\mathbf{e}_2)$ it is possible to take $x_1 = 0$ and so $x_2 \neq 0$. Furthermore, conjugating both generators by $(-I, \mathbf{0})$ if necessary, choose $x_2 > 0$. Now the relation $aba = b$ on the translation components implies that $y_2 = x_2$ since $n \neq 0$. a, b^{-1} form a generating set for Γ, so switching b with b^{-1}

if necessary, assume further that $y_1 > 0$. Therefore generators for this family of homomorphisms are given by

(3) $$a = (A_1, x\mathbf{e}_2), \qquad b = (B_1, y\mathbf{e}_1 + x\mathbf{e}_2)$$

where $x, y > 0$. Using the method outlined above, it can be checked that this action is free on $\mathbb{R}^2$. The essential aspect of these calculations is that both x and y are different from 0. The induced integral affine structure on $\mathbb{T}^2$ is given by

$$a = (A_1, x\mathbf{e}_2), \qquad b^2 = (I, 2y\mathbf{e}_1)$$

where $x, y > 0$ as above.

Case $A = A_2$, $B = B_1$. The group relation on translation components implies that $x_1 = 0 = y_1$. However this makes the translation component of b^2 equal to the zero vector and this is a contradiction. Hence there are no admissible generating sets of this type, as the action is not free.

Remark 3.1. It is interesting to note that while the lower triangular case for A cannot happen for K^2, it can for $\mathbb{T}^2$. Let c, d denote generators for the fundamental group of $\mathbb{T}^2$. [13] shows that if the linear part of one of the generators is not diagonal, then the generating set is conjugate to one of the form

$$c = \left(\begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ w \end{pmatrix} \right), \qquad d = \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} z \\ 0 \end{pmatrix} \right)$$

where $p \in \mathbb{Z}$, $p, w, z > 0$. Conjugate c, d by the reflection

$$\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \mathbf{0} \right)$$

so that the linear part of c is in lower triangular form. This is tantamount to reordering the basis with respect to which we consider the integral affine transformations c, d. The same argument in the K^2 case does not work because conjugation by the reflection above swaps the 1 and -1 eigenspaces. This asymmetry is reflected geometrically in the fact that the above action is shown to be never free.

Case $A = I$, $B = B_2$. Performing transformations as above, assume that the generating set is of the form

(4) $$a = (I, x(\mathbf{e}_1 - \mathbf{e}_2)), \qquad b = (B_2, y\mathbf{e}_2)$$

where $x, y > 0$. This action is also seen to be free and so it induces an integral affine structure on K^2. The induced action on $\mathbb{T}^2$ is given by

$$a = (I, x(\mathbf{e}_1 - \mathbf{e}_2)), \qquad b^2 = (I, y(\mathbf{e}_1 + \mathbf{e}_2))$$

where $x, y > 0$.

Case $A = A_3$, $B = B_2$. The relation $aba = b$ on the translation components implies that

$$\begin{pmatrix} 1+n & n \\ 1-n & -n \end{pmatrix} \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix} = \mathbf{0}$$

Since the determinant of the above matrix is $-2n$ and $n \neq 0$, it follows that $x_1 = -x_2$ and $y_1 = -y_2$. But if this is the case the vector component of b^2 is $\mathbf{0}$ and so the action is not free.

Case $A = A_4$, $B = B_2$. It is convenient to conjugate the generating set by the matrix

$$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

so that the linear parts of a and b become

$$(5) \qquad\qquad A_1 = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}, \qquad B_3 = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$$

As above, let $\mathbf{x}$ and $\mathbf{y}$ denote the vector components of the affine transformations a and b. Conjugating the homomorphism by $(I, -\frac{x_1}{n}\mathbf{e}_2)$, assume that $x_1 = 0$. Switching to a^{-1} if necessary, take $n > 0$. Furthermore, conjugating by $(-I, \mathbf{0})$ if necessary, assume that $x_2 > 0$. Applying the relation $aba = b$ to the vector components, the resulting generating sets are given by

$$(6) \qquad\qquad a = (A_1, x\mathbf{e}_2) \qquad b = (B_3, y\mathbf{e}_1 + \frac{n-1}{n}x\mathbf{e}_2)$$

where $x > 0$ and $y \in \mathbb{R}$ is arbitrary. However, a necessary condition for the action to be free is that $2y \neq -\frac{n-1}{n}x$. The induced integral affine structure on $\mathbb{T}^2$ is given by

$$a = (A_1, x\mathbf{e}_2), \qquad b^2 = (I, (2y + \frac{n-1}{n}x)\mathbf{e}_1)$$

where $x > 0$ and $2y \neq -\frac{n-1}{n}x$.

Remark 3.2. Some of the homomorphisms given by equation (6) are conjugate to maps of equation (3), as pointed out to the author by Ivan Kozlov. In particular, a generating set of equation (6) is conjugate to a homomorphism of equation (3) if and only if the linear part of a takes the form

$$\begin{pmatrix} 1 & 2l+1 \\ 0 & 1 \end{pmatrix}$$

for $l \in \mathbb{Z}$. Homomorphisms belonging to the family

$$(7) \qquad\qquad a = \left(\begin{pmatrix} 1 & 2n \\ 0 & 1 \end{pmatrix}, x\mathbf{e}_2 \right) \qquad b = (B_3, y\mathbf{e}_1 + \frac{n-1}{n}x\mathbf{e}_2)$$

cannot be conjugate to any of the homomorphisms of equations (2) and (4) because of the classification of integral affine structures on $\mathbb{T}^2$ carried out in [13].

Thus there are four families of integral affine structures on K^2. Families with distinct conjugacy classes of generating sets are not conjugate to one another and so integral affine structures on K^2 given by generating sets of different families are not isomorphic. The following theorem summarises the results of this section.

Theorem 3.1. *The only possible homomorphisms $\Gamma \to \mathrm{Aff}_{\mathbb{Z}}(\mathbb{R}^2)$ inducing integral affine structures on K^2 are the ones shown in equations (2), (3), (4) and (7). Integral affine structures arising from distinct families are not be isomorphic to one another as integral affine structures.*

The above classification of integral affine Klein bottles gives insight into their geometry as well. Recall from section 2 that integral affine structures on a manifold B are as flat, torsion-free connections with discrete linear holonomy (lying in

$\mathrm{GL}(n, \mathbb{Z})$). If the linear holonomy consists of orthogonal matrices, then the connection is Levi-Civita. [10] shows that there is only one affine structure on the Klein bottle which induces a Levi-Civita connection up to affine diffeomorphism. The families given by equations (2) and (4) are not integral-affinely isomorphic and both induce a Levi-Civita connection on K^2. This should also be compared with the $\mathbb{T}^2$ case, where there is only one such family. The reason is that, up to conjugation in $\mathrm{GL}(2, \mathbb{Z})$, there are two distinct integral affine involutions on $\mathbb{T}^2$.

4. Explicit Constructions

This section carries out the explicit constructions of Lagrangian fibrations whose base space is the Klein bottle K^2. The methods are similar to those in [13, 14]. The construction is naturally split in two parts: topological and symplectic.

4.1. **Topological Constructions.** As shown in Section 2, the only topological invariants of Lagrangian fibrations are the monodromy and Chern class. The classification of integral affine structures on K^2 given by Theorem 3.1 gives all possible types of linear monodromy that can arise. It is important to bear in mind that given that the monodromy of a Lagrangian fibration is given by taking the inverse transpose of the corresponding linear monodromy of the base space (as an integral affine manifold. Set

$$A_2 = \begin{pmatrix} 1 & 2n \\ 0 & 1 \end{pmatrix}.$$

The monodromy of a Lagrangian fibration over the Klein bottle is then given by one of the following four families below

$$\rho_i(a) = \begin{cases} I & \text{if } i = 1; \\ (A_1^{-1})^T & \text{if } i = 2; \\ I & \text{if } i = 3; \\ (A_2^{-1})^T & \text{if } i = 4. \end{cases} \qquad \rho_i(b) = \begin{cases} (B_1^{-1})^T & \text{if } i = 1; \\ (B_1^{-1})^T & \text{if } i = 2; \\ (B_2^{-1})^T & \text{if } i = 3; \\ (B_3^{-1})^T & \text{if } i = 4. \end{cases}$$

where A_1 and B_1, B_2, B_3 are defined as in Section 3 and A_2 is defined as above. For each such monodromy representation ρ_i, there are Chern classes given by elements in the twisted cohomology group $\mathrm{H}^2(K^2; \mathbb{Z}^2_{\rho_i})$. It follows from [6, 16] that *all* elements of $\mathrm{H}^2(K^2; \mathbb{Z}^2_{\rho_i})$ arise as the Chern class of some Lagrangian fibration with monodromy ρ_i. However, this result is also proved by means of explicit constructions below.

The next lemma computes the twisted cohomology groups $\mathrm{H}^2(K^2; \mathbb{Z}^2_{\rho_i})$. Let n be the non-diagonal entry of A_1.

Lemma 4.1. *The twisted cohomology groups* $\mathrm{H}^2(K^2; \mathbb{Z}^2_{\rho_i})$ *are given by*

$$\mathrm{H}^2(K^2; \mathbb{Z}^2_{\rho_i}) = \begin{cases} \mathbb{Z}/2 \oplus \mathbb{Z} & \text{if } i = 1; \\ \mathbb{Z}/2 \oplus \mathbb{Z}/n & \text{if } i = 2; \\ \mathbb{Z} & \text{if } i = 3 \\ \mathbb{Z}/4n & \text{if } i = 4. \end{cases}$$

Proof. These calculations follow from standard methods in algebraic topology [5]. The standard cell decomposition of K^2

$$K^2 = e^0 \cup e_1^1 \cup e_2^1 \cup e^2$$

induces a Γ-equivariant cell decomposition on its universal cover $\tilde{K}^2 = \mathbb{R}^2$, given by

$$\mathbb{R}^2 = \bigcup_{\alpha \in \Gamma} (e_\alpha^0 \cup e_{1,\alpha}^1 \cup e_{2,\alpha}^1 \cup e_\alpha^2)$$

As a $\mathbb{Z}[\Gamma]$-basis for the module $C_i(\tilde{K}^2)$ take the following elements

$$\begin{cases} e_{\alpha_0}^0 & \text{if } i = 0 \\ e_{1,\alpha_0}^1, e_{2,\alpha_0}^1 & \text{if } i = 1 \\ e_{\alpha_0}^2 & \text{if } i = 2 \end{cases}$$

for some fixed element $\alpha_0 \in \Gamma$. In order to simplify notation, forget about the α_0 dependence. With respect to the above basis the following $\mathbb{Z}[\Gamma]$-equivariant chain complex

$$0 \longrightarrow C_2(\tilde{K}^2) \xrightarrow{\partial_2} C_1(\tilde{K}^2) \xrightarrow{\partial_1} C_0(\tilde{K}^2) \longrightarrow 0$$

is defined by maps ∂_2, ∂_1 which are given on the basis elements by

$$\begin{aligned} \partial_2 e^2 &= (1+b)e_1^1 + (a-1)e_2^1 \\ \partial_1 e_1^1 &= (a-1)e^0 \\ \partial_1 e_2^1 &= (ba-1)e^0 \end{aligned}$$

Using the $\mathbb{Z}[\Gamma]$-equivariant Hom functor, the following $\mathbb{Z}[\Gamma]$-equivariant cochain complex arises

$$0 \longrightarrow \mathrm{Hom}(C_0(\tilde{K}^2); \mathbb{Z}^2) \xrightarrow{\delta_1} \mathrm{Hom}(C_1(\tilde{K}^2); \mathbb{Z}^2) \xrightarrow{\delta_2} \mathrm{Hom}(C_2(\tilde{K}^2); \mathbb{Z}^2) \longrightarrow 0$$

Since a $\mathbb{Z}[\Gamma]$-equivariant homomorphism $C_2(\tilde{K}^2) \to \mathbb{Z}^2$ ($\mathbb{Z}^2$ is a $\mathbb{Z}[\Gamma]$-module via the representations ρ_i for a fixed i) is completely determined by its value on the basis element e^2, the following holds

$$H^2(K^2; \mathbb{Z}_\rho^2) \cong \mathbb{Z}^2 / \mathrm{im}\, \delta_2$$

There are four cases to consider. Explicit calculations are given for the case $i = 1$, all other ones are similar in spirit and therefore omitted.

Case ρ_1: Let $\phi : C_1(\tilde{K}^2) \to \mathbb{Z}^2$ be an equivariant homomorphism, then

$$(\delta_2(\phi))(e^2) = \phi(\partial_2 e^2) = (I + \rho_1(b))\phi(e_1^1) + (\rho_1(a) - I)\phi(e_2^1)$$

$$= \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \phi(e_1^1) = \begin{pmatrix} 2s \\ 0 \end{pmatrix}$$

where $\phi(e_1^1) = (s, t) \in \mathbb{Z}^2$. Since the choice of s is arbitrary, $\mathrm{im}\, \delta_2 \cong 2\mathbb{Z} \oplus 0$ and we get that

$$H^2(K^2; \mathbb{Z}_{\rho_1}^2) \cong \mathbb{Z}/2 \oplus \mathbb{Z}$$

Case ρ_2: In this case, $\mathrm{im}\, \delta_2 \cong 2\mathbb{Z} \oplus n\mathbb{Z}$, so that

$$H^2(K^2; \mathbb{Z}_{\rho_2}^2) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/n$$

Case ρ_3: Following the methods above, $\mathrm{im}\, \delta_2 \cong \{(q, q) : q \in \mathbb{Z}\} \cong \mathbb{Z}$, so that

$$H^2(K^2; \mathbb{Z}_{\rho_3}^2) \cong \mathbb{Z}$$

Case ρ_4: In this case, $\mathrm{im}\, \delta_2 \cong \mathbb{Z} \oplus 4n\mathbb{Z}$ and so $H^2(K^2; \mathbb{Z}_{\rho_4}^2) \cong \mathbb{Z}/4n$.

$\square$

The following shows how to construct smoothly all Lagrangian fibrations over K^2. Take the trivial fibration $(\mathbb{R}^2 \times \mathbb{T}^2, \omega_0) \to \mathbb{R}^2$ and, for a given affine monodromy homomorphism $\tau_i : \Gamma \to \mathrm{Aff}_{\mathbb{Z}}(\mathbb{R}^2)$ whose linear part is denoted by ρ_i, define a group action of Γ on $\mathbb{R}^2 \times \mathbb{T}^2$ by

$$\tau \cdot (\mathbf{x}, \mathbf{t}) = (\tau_i \cdot \mathbf{x}, \rho_i^* \cdot \mathbf{t})$$

where ρ_i^* denotes the inverse transpose of ρ_i. The above transformations are bundle automorphisms and they preserve the symplectic form ω_0. Hence the quotient bundle $(\mathbb{R}^2 \times \mathbb{T}^2)/\Gamma \to K^2$ is a Lagrangian fibration with linear monodromy given by ρ_i. Furthermore these fibrations have a global section (the image of the zero section); this therefore constructs all possible topological reference Lagrangian fibrations over K^2. Fix an affine monodromy representation $\tau_i : \Gamma \to \mathrm{GL}(2, \mathbb{Z})$ and think of K^2 as a square with opposite sides appropriately identified. Let $\pi : (M, \omega) \to K^2$ be the Lagrangian fibration with linear monodromy given by ρ_i and zero Chern class. Cut out $\pi^{-1}(D_{2\epsilon})$ from M, where $D_{2\epsilon}$ is a small closed disc of radius 2ϵ in the interior of the square. In a neighbourhood of this disc the symplectic form is given by

$$\omega = \mathrm{d}x \wedge \mathrm{d}t_1 + \mathrm{d}y \wedge \mathrm{d}t_2$$

where x, y are local coordinates on a neighbourhood of $D_{2\epsilon}$ in K^2 and t_1, t_2 are coordinates on the fibre. Define a new fibration $M' \to K^2$ by

$$M = (M - (\mathrm{int}\,\pi^{-1}(D_\epsilon))) \cup_h \pi^{-1}(D_{2\epsilon})$$

where the attaching map h is defined on the intersection as

$$(x, y, t_1, t_2) \mapsto \left(x, y, t_1 + \frac{m}{2\pi}\arg(x + \iota y) - \frac{n}{4\pi}f(x, y), t_2 + \frac{n}{2\pi}\arg(y + \iota x) - \frac{m}{4\pi}f(y, x)\right)$$

where $f(x, y) = \frac{1}{2}\log\left(\frac{x^2 + y^2}{1 + y^2}\right)$ and arg is a well-defined map since it maps into an S^1 copy of the fibre $F \cong \mathbb{R}^2/\mathbb{Z}^2$. Since h is a symplectomorphism (this can be checked explicitly by considering the pullback of ω under h), the new bundle is still Lagrangian and has Chern class depending on the pair $(m, n) \in \mathbb{Z}^2$; some pairs (m, n) will not change the topological structure of the original bundle, as the resulting bundle still admits a global section. However, it is clear from the construction that *all* Chern classes in $\mathrm{H}^2(K^2; \mathbb{Z}^2_{\rho_i})$ can be realised and that the Chern class of the bundle constructed above is the pair (m, n) modulo appropriate trivial pairs.

The above discussion proves the following

Theorem 4.1. *The linear monodromy representations ρ_i and the elements of the corresponding twisted cohomology groups $\mathrm{H}^2(K^2; \mathbb{Z}^2_{\rho_i})$ classify regular Lagrangian fibrations with base space K^2 up to topological bundle isomorphism. For a fixed linear monodromy representation,* all *elements of the corresponding cohomology group can be realised as the Chern class of some regular Lagrangian fibration.*

4.2. **Symplectic Constructions.** It remains to construct all possible regular Lagrangian fibrations over K^2 up to fibrewise symplectomorphism covering the identity on the base space. As shown in sections 1 and 2, there is no loss of generality in restricting to this this type of symplectomorphisms; furthermore, for this class of maps there are no issues with explicit choices of period-lattice bundles as shown in section 2. Fix an integral affine structure belonging to one of the families of Theorem 3.1. As seen in the previous section, this fixes the monodromy representation

of the regular Lagrangian classes over K^2 inducing this integral affine structure. Fix the topological type of such a Lagrangian fibration, *i.e.* fix the Chern class. The space of Lagrangian classes for this fibration is a quotient of $\mathrm{H}^2(K^2; \mathbb{R})$. Since $\mathrm{H}^2(K^2; \mathbb{R}) = 0$ this space is trivial. This discussion proves the following

Theorem 4.2. *For a fixed regular Lagrangian fibration over K^2 with given linear monodromy ρ_i ($i = 1, \ldots, 4$) as in Theorem 3.1 and fixed Chern class, all symplectic structures preserving Lagrangianeity of this fibration are fibrewise symplectomorphic over the identity.*

This result is analogous to the main result in [13]. Fix an integral affine structure on $\mathbb{T}^2$ and the corresponding topologically reference Lagrangian fibration. The explicit choice of translation components for the generating set determines the explicit form of the subgroup H of $\mathrm{H}^2(\mathbb{T}^2; \mathbb{R})$ which acts trivially on the space of symplectic structures preserving Lagrangianeity of the fibration, as shown in [13]. Contrast this with the K^2 case. Lagrangian fibrations over K^2 inducing the same linear monodromy but distinct translation components are not going to be fibrewise symplectomorphic over the identity; however, the explicit choice of translation components does not matter when determining the space of Lagrangian classes for fixed linear monodromy as it is trivial in all cases.

5. Conclusion

This completes the classification of Lagrangian fibrations whose base space is K^2. However, much more is yet to be known about this type of fibrations in general. Even the case when the base is a non-compact connected 2-surface is interesting to consider. Furthermore, a classification of singular Lagrangian fibrations over surfaces would be the natural next step to take. There exist examples of such singular fibrations, like the famous 24 focus-focus singularities fibration over the sphere S^2 as mentioned in [16]. Better insight into this kind of singular fibrations would be of interest and use to a wide range of mathematicians, ranging from dynamicists to algebraic geometers and people interested in mirror symmetry.

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